A R DIGITECH
International Journal Of Engineering, Education And Technology (ARDIJEET)
www.ardigitech.in ISSN 2320-883X,VOLUME 10 ISSUE 03 15/08/2022

From Symmetry to Complexity: Self Complementary Graphs Unraveled

Shivam Sharma, Department of computer science, R D Engineering College, Ghaziabad
Corresponding author-shivam.sharma@gmail.com

Abstract

Self-complementary graphs are an intriguing and significant subject in graph theory. One special quality of these graphs is that they are isomorphic to their own complement. Self-complementary graphs are of tremendous interest to mathematicians, computer scientists, and researchers in other fields because of their fascinating property. The goal of this study is to provide an overview of the structure and features of the class of self-complementary graphs. Directions for future work are also presented.

Keywords: Complement of a graph, Isomorphic graphs, Self-complementary graphs.

1. Introduction

Graphs that are self-complementary are frequently very symmetric. They can exhibit reflectional and rotational symmetry, among other kinds of symmetry. Mathematicians and graph theorists find self- complementary graphs to be extremely fascinating because of this symmetry. It was the seminal publications of Ringel [1] and Sachs [2] that launched the field of study on self-complementary graphs. The subject was born under fortunate circumstances, and Rao [3] and Bosak [4] have traced its evolution over the last four decades. Additionally, it has been the subject of four Ph.D. theses, at least partially [5, 6, 7, 8], and four more that address related subjects [9, 10, 11, 12], in addition to the hundreds of papers that have been published so far. In the discipline of graph theory, self-complementary graphs are a fascinating and significant subject. One special quality of these graphs is that they are isomorphic to their own complement. Stated otherwise, a self-complementary graph is one that is isomorphic to the original graph when its complement is taken (that is, a graph where neighboring vertices in the original graph are non-adjacent, and vice versa). Self-complementary graphs are of tremendous interest to mathematicians, computer scientists, and researchers in other fields because of their fascinating property.

William Tutte studied the idea of self-complementary graphs in the middle of the 20th century. Tutte's research on this subject served as a basis for numerous other graph theory projects. Additionally, self- complementary graphs have a restricted structure. Sachs [2] pointed out that the permutations of a self-complementary graph's vertices that map it into its complement have an intriguing cycle structure. One may wonder how many self-complementary graphs there are on n vertices for a given $n \in N$. In a study [13] that was published in 1963, R. C. Read lists these values for arbitrary $n \in N$. The enumeration reveals the interesting fact that the number of digraphs with $2n$ vertices that are self-complementary digraphs is equal to the number of self-complementary graphs with $4n$ vertices for every $n \in N$. In 1975, D. Wille found a similar result. He established the following: the number of self-complementary relations over $2n$ components are equal to the number of self-complementary graphs with $4n + 1$ vertices [14].

A natural bijection between the self-complementary graphs with $4n$ vertices and the self-complementary digraphs with $2n$ vertices is probably present, according to Read's assertion in 1987. This indicates that, for all $n \in N$, there is a general process that transfers the self-complementary digraphs with $2n$ vertices onto the self-complementary graphs with $4n$ vertices. B. Zelinka has made one attempt to identify a natural bijection [19]. Unfortunately, the only image available for the paper was a poor quality one, making it impossible to rebuild several important symbols like indices. The algorithm provided by B. Zelinka is not well-defined because the complementing permutations it is built on are not unique. Numerous mathematicians have attempted, but failed, to discover such a natural bijection. As far as we are aware, self-complementary vertex-transitive graphs seem to be rather uncommon and challenging to create. Several authors have discovered methods to construct it [20, 21]. We present some of them in the third section of this article. The goal of this paper is to give references to the substantial body of research that has been conducted and to present a few precise findings. Whenever feasible, proofs are also included to pique the reader's interest.

2. Definitions and Preliminaries

In this section we give the basic definitions and some properties of self-complementary graphs. We use the notation sc-graph for self-complementary graphs. For all basic terminologies and notations not mentioned here, we refer to [15].

Definition 1: The complement G' of a graph G has the same vertices as G and every pair of vertices are joined by an edge in G' if and only they are not joined in G (Figure 1).

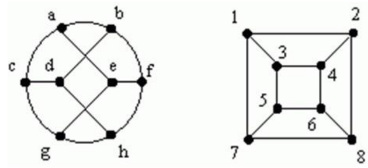

Figure 1: G and its complement G'

Definition 2: A self-complementary graph is one that is isomorphic to its complement G'.

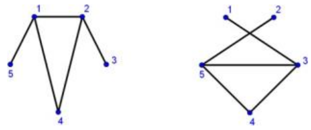

Figure 2: A self-complementary graph

Definition 3: Degree of a vertex is the number of vertices to which it is adjacent.

Definition 4: The degree sequence of a graph is the sequence of degrees of all its vertices arranged in non- increasing order $d_1 \geq d_2 \geq d_3 \geq \ldots$

Definition 5: An isomorphism f from G to H is a bijection between its vertex sets, $f: V(G) \rightarrow V(H)$
such that any 2 vertices u and v of G are adjacent in G iff $f(u)$ and $f(v)$ are adjacent in H

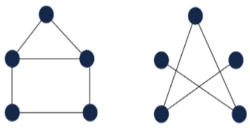

Figure 3: Isomorphic graphs

Definition 6: An isomorphism from a graph to its complement is called an anti-morphism.

Definition 7: The distance between two vertices v and w, denoted by $d(v, w)$, is the length of a shortest path between them, or ∞ if there is no such path. Thus, $d(v, v) = 0$, while $d(v, w) = 1$ if and only if v and w are adjacent.

Definition 8: The eccentricity of a vertex v is the maximum of all distances $d(v, w)$. The diameter [radius] of a graph G is the maximum [minimum] of the eccentricities of all vertices of G, and is denoted by $diam(G)[rad(G)]$. Thus, G is disconnected if and only if $diam(G) = rad(G) = \infty$; connected graphs have finite radius and diameter.

Definition 9: An edge vw of a graph G is a dominating edge of G if all vertices of G are adjacent to either v or w, or both.

Definition 10: A vertex v in a connected graph G is called a cut-vertex if $G - v$ is disconnected. A connected graph is said to be k-connected if the removal of less than k vertices leave a subgraph that is still connected. Thus, if G has cut-vertices it is only 1-connected; if it does not have cut-vertices, it is (at least) 2-connected and is called a block.

Definition 11: A vertex of degree one is called an end-vertex, and the number of end-vertices in G is denoted by $v_1(G)$.

Definition 14: A regular graph is one in which all vertices are of same degree. An almost regular graph is a graph whose vertices have one of two degrees, s and $s-1$ for some s; we insist that at least one vertex has degree s, and at least one vertex degree $s-1$, otherwise, we get a regular graph.**Theorem 2:** G and G' cannot both be disconnected.

Theorem 3: All self-complementary graphs are connected.

Theorem 4: [22] A self-complementary graph G has cut-vertices if and only if it has end-vertices.

Proof: Since K_2 is not self-complementary, any sc-graph with end-vertices must have cut-vertices. Now, if G is a sc-graph with cut-vertex v, but no end-vertices, then $G - v$ has at least two components. Let one of these components be A and let $G - v = A \cup B$. Then $(G-v)'$ contains a spanning bipartite subgraph, with parts A and B. Since G has no end-vertices, A and B each have cardinality at least two; and v has degree at least 2 in G, and thus also in G. But then G is 2-connected, a contradiction.

Theorem 5: [22] If a graph G has at least two end-vertices, then G has at most two end-vertices. **Proof:** Let v and w be two end-vertices of G, adjacent to x and y. Then the only candidates for end- vertices in G are x and y, as all other vertices have degree at most $n-3$ in G.

Theorem 6:[22] A vertex v has eccentricity at least 3 in G if and only if, in G', it lies on a dominating edge and has eccentricity at most 2.

Proof: If $v \in G$ has eccentricity at least 3, then $d(v,w) \geq 3$, for some $w \in G$. As a result, v and w are non-adjacent, and they do not have a common neighbour. Then, v and w are neighbours in G', and every other vertex will be adjacent to v, w, or both. Hence, v has eccentricity of no more than two, and vw is a dominating edge of G. Converse is clear.

Theorem 7: [22]For any graph G the following hold:

(i) If $rad(G) \geq 3$ then $rad(G') \leq 2$.

(ii) $Diam(G) \geq 3$ if and only if G has a dominating edge.

(iii) If $diam(G) \geq 3$ then $diam(G) \leq 3$.

(iv) If $diam(G) \geq 4$ then $diam(G) \leq 2$.

Theorem 8: [22] Let G be a non-trivial self-complementary graph; then

(i) G has radius 2 and diameter 2 or 3.

(ii) G has diameter 3 if and only if it contains a dominating edge.

(iii) The number of vertices of eccentricity 3 is never greater than the number of vertices of eccentricity 2.

Theorem 9: [22]A regular self-complementary graph G must have $4k + 1$ vertices and degree $2k$ for some k and diameter 2. An almost regular sc-graph must have $4k$ vertices, of which half have degree $2k$ and half $2k - 1$, for some k. Moreover, the regular and almost regular sc-graphs are in one-one correspondence.

Definition 6: A graph is called planar if it can be embedded in the plane.

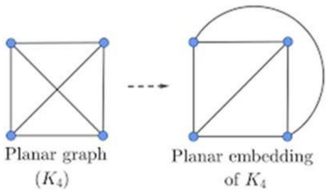

Figure 5: Planar graph and its planar embedding

Definition 7: A graph is called outer planar if it can be embedded in the plane such that all its vertices sit on the outer face of the embedding.

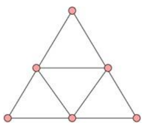

Figure 6: An outer planar graph

Theorem 10: [22] A self-complementary graph on n≥8 vertices is not outer planar and for n≥9, it is not planar.

Definition 15: The minimum number of colors required to color a graph properly is called the chromatic number

Theorem 11: [22] Let G be a self-complementary graph. Then $\sqrt{n} \leq \chi \leq (n + 1)/2$, where χ is the chromatic number of G. In particular, for any constant r, there are only a finite number of self- complementary graphs which are r-partite.

Definition 16: A graph is said to be perfect if for every induced subgraph the chromatic number equals the clique number.

Obviously neither a perfect graph nor its complement can contain an induced odd circuit of size at least

5. Berge has conjectured that the converse is true.

Strong Perfect Graph Conjecture:[16] A graph G is perfect if and only if neither G nor G' contains an induced odd cycle of size at least 5.

Theorem 12:[16] The Strong Perfect Graph Conjecture is true in general if and only if it is true for (bi regular) sc -graphs.

Theorem 13: [3] If G is a self-complementary graph on n > 5 vertices, then for every integer $3 \leq l \leq$

$n - 2$, G contains a circuit of length l. Further, if G is Hamiltonian then it is pancyclic.

Self-complementary graphs are useful tools for studying Ramsey numbers. The existence of regular self- complementary graphs of order n is well known and can be demonstrated with ease if and only if $n \equiv 1 \ (mod\ 4)$. Thus, it seems sense to inquire as to whether self-complementary vertex-transitive graphs can have a comparable outcome. More specifically, for which positive integers n, do self-complementary vertex-transitive graphs of order n exist is a question of great interest. Though the general question remains unanswered, a few long-standing outstanding issues pertaining to this question have recently been resolved.

3. Construction of self-complementary graphs

Method of construction of self-complementary graphs were put forward by several authors [17,18]. We present one of the methods here. This construction is illustrated in Figure 7.

Step 1: Let H be any graph which is self-complementary, and let $P_4 = v_1v_2v_3v_4$ be a 4-path, i.e., a path with exactly four vertices.

Step 2: Join each of v_2 and v_3 to all vertices of H. We call this operation a 4-path addition.

Illustration

Let H be C_5. After the 4-path addition operation, we get a self-complementary graph of 9 vertices.

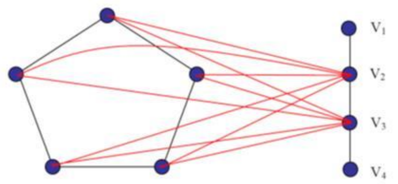

Figure 7: Illustration of method of construction of sc-graph

4. Applications

Network design, error-correcting codes, and coding theory are among the fields in which self- complementary graphs are used. They are also employed in computer science fields where graph structures are important, such as encryption. The creation of more effective data structures and algorithms can result from an understanding of the characteristics of self-complementary graphs.

5. Scope for further research

A few directions of future research are as follows:

- Can a self-complementary graph G be co-spectral to a graph H which is not self-complementary?
- If such a graph exists, must it also be co-spectral to H?
- When can a self-complementary graph G be co-spectral to another self-complementary graph?

Self-complementary graphs continue to be a subject of active research. There are still open questions about the existence and classification of self-complementary graphs for various orders. Researchers are also exploring the relationship between self-complementary graphs and other graph classes, such as strongly regular graphs.

6. Conclusion

To sum up, self-complementary graphs are an interesting and significant field of graph theory research. Researchers in several fields, including mathematicians and computer scientists, are interested in them because of their special trait of being isomorphic to their complement. New methods and insights in graph theory have been developed as a result of the study of self-complementary graphs, and these developments may find use in the fields of network architecture, encryption, and coding theory. More findings and uses of self-complementary graphs are probably in store as this area of study develops, underscoring its importance in the fields of mathematics and computer science.

7. References

1. G. Ringel, Selbstkomplementare Graphen, Arch. Math. 14 (1963) 354– 358.
2. H. Sachs, Uber selbstkomplementare graphen, Publ. Math. Drecen 9 (1962) 270–288.
3. S.B. Rao, Explored, semi-explored and unexplored territories in the structure theory of self- complementary graphs and digraphs, in Proc. Symposium on Graph Theory, Indian Statist. Inst., Calcutta, (1976). ISI Lecture Notes, MacMillan, India, 4 (1979a) 10–35.
4. J. Bosak, Decompositions of graphs, Kluwer Academic Publishers, Dordrecht/ Boston/ London, Mathematics and Its Applications (East European Series) 47 (1990) xvii+248 pp..
5. L.D. Carrillo, On strongly Hamiltonian self-complementary graphs, Ph.D. Thesis, Ateneo de Manila Univ., Philippines (1989).
6. T. Gangopadhyay, Studies in multipartite self-complementary graphs, Ph.D. Thesis, I.S.I. Calcutta (1980).
7. R.A. Gibbs, Self-Complementary Graphs:

Their Structural Properties and Adjacency Matrices, Ph.D. Thesis, Michigan State University (1970).

8. B.R. Nair, Studies on triangle number in a graph and related topics, Ph.D. Thesis, School of Mathematical Sciences, Cochin University of Science and Technology, India (1994) 96 pp.
9. M. Hegde, Some enumerative digraph and hypergraph problems, Ph.D. Thesis, I.I.T. Kanpur, India (1978).
10. S.J. Quinn, On the existence and enumeration of isomorphic factorizations and graphs, Master's Thesis, University of Newcastle, Australian (1984).
11. S. Ruiz, On Isomorphic Decompositions of Graphs, Ph.D. Thesis, Western Michigan University, Kalamazoo (1983).
12. D. Wille, Asymptotische Formeln fÄur Strukturzahlen, Dissertation, Technological University Hannover (1971).
13. R. C. Read On the number of self-complementary graphs and digraphs J. London Math. Soc. 38 (1963) 99-104.
14. D. Wille, Enumeration of self-complementary structures Journal of combinatorial theory, series B 25 (1978) 143-150.
15. D.B. West, Introduction to graph theory, Pearson Education Inc., Delhi, (2001).
16. D.G. Corneil, Families of graphs complete for the strong perfect graph conjecture, J. Graph Theory 10 (1986) 33-40.
17. J. Xu, C.K. Wong, Self-complementary graphs and Ramsey numbers Part I: the decomposition and construction of self-complementary graphs, Discrete Mathematics 223 (2000) 309-326.
18. L. Wang, C. H. Li, Y. Liu, C. X. Wu, New constructions of self-complementary Cayley graphs; The Electronic Journal of Combinatorics 24(3) (2017),3-19.
19. B. Zelinka, Self-complementary vertex-transitive undirected graphs, Math. Slovaca 29 (1979) 91–95.
20. H. Zhang, Self-complementary symmetric graphs, J. Graph Theory 16 (1992) 1–5.
21. C. H. Li, On Finite Graphs That Are Self-complementary and vertex-transitive; Australasian Journal of Combinatorics 18(1998),pp.147-155
22. A. Farrugia, Self-complementary graphs and generalizations: a comprehensive reference manual, Master’s Thesis, University of Malta,1999.