Selection of Adaptive Wavelet for Wavelet Transforms in Image Processing

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Abstract: Wavelet transform is an important analytical tool because it captures localized information of a signal or image. Multiresolution analysis is a recursive method which provides flexibility in the analysis of signals/data. The selection of wavelet for wavelet transforms in image processing is primary and most important work because it decides the accuracy of results. In wavelet transforms the image is divided into four sub-bands after one level decomposition in which the left upper part represents the approximated version of image and contains coarse information of the image. The total size or length of all vectors in a vector space or matrix is referred to as the norm. The L^1 *and* L^2-norms of the given image represent compactness of the energy and hence provide a base for the selection of best adaptive wavelet.

Keywords: Approximation, image, L^1 *and* L^2-norms, wavelet.

1. Introduction

Fourier transforms is an important analytical tool and widely used in science and engineering for long times. It is well suitable for stationary signals but not suitable for non-stationary and transient signals. Non-stationary is taken as combination of small stationary signals and analysed with help of Window Fourier transforms (WFT) or short time Fourier transforms (STFT). In STFT signal is multiplied by a small sized window function and its size is restricted by Heisenberg uncertainty principle. Its time and frequency resolution both are not good at the same time. To overcome the discrepancies of resolution of STFT, the wavelet theory is proposed in 1980s [1]. Wavelet is a new and favourable tool used to analyse the signals especially nonstationary and transient signals. A wavelet is a fast-decaying oscillatory waveform which can be translated and dilated. With help of wavelet a signal or mathematical function can be divided into different scale components [2]. A wavelet is an oscillating function or small wave of zero average and well localized over a short time interval. The family of wavelets is a set of wavelet functions that can be obtained by translating (shifting) and dilating (stretching or compressing) an original function called mother wavelet. Wavelets are used to encode the data, looking for heart abnormalities, in computing, imaging and animation, predicting the weather etc. [3]. Wavelet transform is related to harmonic analysis because of their time-frequency representation for analogue and discrete signals. In wavelet transforms the scaling and wavelet coefficients are called filter banks [4].

Selection of adaptive wavelet is the primary and most important work in the field of signal processing using wavelet transforms. We have proposed a technique for selecting the wavelet which is best adaptive for the wavelet transforms in the image processing.

2. Basics of wavelet transforms

Wavelet is a small wave of zero average, which can be dilated and translated. A whole set of wavelets can be obtained by translating and scaling the mother wavelet as follows: -

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) = T_b D_a \psi \qquad (1)$$

Here a and b are the dilation and translation parameters respectively.

Continuous wavelet transform of any function provides high amount of information about this function and can be expressed in terms of two parameters a and b as follows: -

$$W_{a,b} = \int f(x) \frac{1}{\sqrt{a}} \phi\left(\frac{x-b}{a}\right) dx$$

$$= \int f(x)\, \phi_{a,b}(x) dx \qquad (2)$$

By introducing $a = 2^{-j}$, and $\frac{b}{a} = k$ where j and k are integers, the discrete wavelet transform is expressed as follows [5]: -

$$W_{j,k} = \int f(x) 2^{j/2}\, \phi(2^j x - k)\ dx \qquad (3)$$

The multiresolution analysis (MRA) is a new, flexible and recursive method of discrete wavelet transforms which is frequently used to analyse the signals. Mallat introduced the idea of MRA consisting of a closed vector subspace $V_{j,}\ j \in \mathbb{Z}$ of square integrable function called Lebesgue space $L^2(\mathbb{R})$ [6-7]. The Lebesgue space satisfies the following properties: -

1) $V_{j+1} \subset V_j : j \in \mathbb{Z}$

2) $\cap_{j\in\mathbb{Z}} V_j = \{0\},\ \cup_{j\in\mathbb{Z}} = L^2(\mathbb{R})$

3) $f(x) \in V_j \Rightarrow f(2x) \in V_{j+1},\ \forall\ j \in \mathbb{Z}$

4) If a function $\phi(x) \in V_0$, then $\{\phi(2x-k) : k \in \mathbb{Z}\}$ is orthonormal basis of $V_{0.}$

Here the function $\phi(x)$ is known as a scaling function of given MRA and can be expressed by a dilation equation as follows [8]: -

$$\phi(x) = \sum_{k\in\mathbb{Z}} \alpha_k\, \phi(2x-k) \qquad (4)$$

where α_k is called a low pass filter and described as follows: -

$$\alpha_k = \left(\frac{1}{\sqrt{2}}\right) \int_{-\infty}^{\infty} \phi(x)\phi(2x-k) dx \qquad (5)$$

If $\psi \in W_0$ be any function then,

$$\psi(x) = \sum_{k\in Z} \beta_k\ \phi(2x-k) \qquad (6)$$

Where, $\beta_k = (-1)^{k+1}\, \alpha_{1-k}$ is high pass filter.

Any signal in vector space V_0 can be expressed in terms of bases of subspaces V_1 and W_1. Therefore, we can express space V_0 in terms of subspaces V_1 and W_1 as follows: -

$$V_0 = V_1 \oplus W_1$$

In general,

$$V_j = V_{j+1} \oplus W_{j+1}$$

But, $V_{j+1} = V_{j+2} \oplus W_{j+2}$

Therefore,

$$V_j = W_{j+1} \oplus W_{j+2} \oplus V_{j+2}$$

.. ..

$$V_j = W_{j+1} \oplus W_{j+2} \oplus W_{j+3} \oplus \ldots\ldots\ldots\ldots W_{j_0} \oplus V_{j_0}$$

Any integrable function f, has the series expansion,

$$f(x) = \sum_{k\in\mathbb{Z}} \langle f,\ \tilde{\phi}_{j_0,k} \rangle\, \phi_{j_0,k}(x) + \sum_{p=j+1}^{j_0} \sum_{k\in\mathbb{Z}} \langle f,\ \tilde{\psi}_{p,k} \rangle\ \psi_{p,k}(x)$$

Similarly, the roles of the basis and the dual basis can be interchanged [9]. Here, $\sum_{k\in\mathbb{Z}} \langle f,\ \tilde{\phi}_{j_0,k} \rangle\, \phi_{j_0,k}(x)$ is a coarse scale V_{j_0} – approximation of f and for every p, the sum $\sum_{k\in\mathbb{Z}} \langle f,\ \tilde{\psi}_{p,k} \rangle\ \psi_{p,k}(x)$, adds the detail in spaces W_p.

3. Image wavelet transforms

A digitization is a process by which any analogue image in continuous space can be converted into a digital image in discrete space through sample process. Let us consider that an analogue image can be divided into M rows and N columns. The value can be assigned to the integer coordinates $[m, n]$ like {m = 0, 1, 2, . . ., $M-1$} and {n = 0, 1, 2, . . ., $N-1$}. In image wavelet transforms, the scaling and wavelet function are described as two variable functions $\phi(x, y)$ and $\psi(x, y)$ [10]. The image scaling and wavelet functions are easily expressed in terms of 1D functions as follows: -

$$\phi(x, y) = \phi(x)\phi(y)$$
$$\psi^1(x, y) = \psi(x)\phi(y)$$
$$\psi^2(x, y) = \phi(x)\psi(y)$$
$$\psi^3(x, y) = \psi(x)\psi(y)$$

Obviously the 2D (Image) functions are described as the superposition of 1D scaling and wavelet functions. The 2D approximation and detail coefficients are determined as follows: -

$$a[j_0, m, n] = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, \phi_{j_0,m,n}(x, y)$$

$$d^i(j, m, n) = \frac{1}{\sqrt{MN}} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\, \psi^i_{j,m,n}(x, y) \text{ where } i = \{1,2,3\}$$

$$f(x, y) = \frac{1}{\sqrt{MN}} \sum_m \sum_n a[j_0, m, n]\, \phi_{j_0,m,n}(x, y)$$
$$+ \frac{1}{\sqrt{MN}} \sum_{i=1,2,3} \sum_{p=j+1}^{j_0} \sum_m \sum_n d^i(p, m, n)\, \psi^i_{p,m,n}(x, y)$$

In 2D-wavelet transforms of an image, the scaling and wavelet functions can be expressed as follows: -

$$\phi_{j,m,n}(x, y) = 2^{j/2}\phi(2^j x - m, 2^j y - n)$$
$$\psi^i{}_{j,m,n}(x, y) = 2^{j/2}\psi^i(2^j x - m, 2^j y - n)$$

for $1 \le i \le 3$. The wavelet functions $\left\{\psi^1{}_{j,m,n}, \psi^2{}_{j,m,n} \psi^3{}_{j,m,n}\right\}$ form an orthonormal basis of the subspace of details and described as follows: -

$$W_j{}^2 = (V_j \otimes W_j)(W_j \otimes V_j) \oplus ((W_j \otimes W_j)$$

at scale j. The Lebesgue space $L^2(\mathbb{R}^2)$ can be expressed as follows: -

$$L^2(\mathbb{R}^2) = \sum_j W_j{}^2$$

The 2D wavelet transform is considered as superposition of two 1D wavelet transforms along X and Y-dimension [11]. In image processing, an image is passed through these filter banks followed by a decimation process. Due to the decimation, the total size of the transformed image is conserved and equal to that of the original image. In image wavelet transforms, the original image is divided into four sub-bands denoted by $LL1$, $HL1$, $LH1$, and $HH1$ in each level of decomposition (Figure 1).

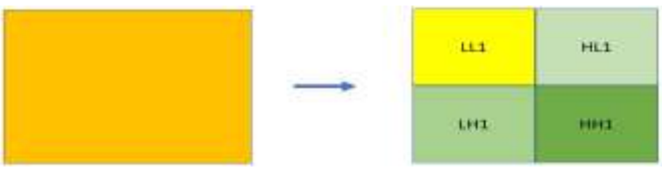

Figure 1: **Decomposition of an image (level-1)**

Here all $LL1$, $HL1$, $LH1$ and $HH1$ are each belongs to $M/2 \times N/2$ submatrices. The approximation $LL1$ contains scaling coefficients occupies the upper left quadrant out of four in the 2D wavelet decomposition. The $LL1$ represents the trend of image and has great importance because it contains the average characteristics of the image.

The wavelet and statistical analysis of sub-image corresponding to trend $LL1$ of original image (face expression) are being done and interpreted.

4. Research methodology

We have selected an image of a girl as sample image. The image wavelet transforms using different wavelets at level 1 are performed. The subband $LL1$ represents approximation of the image at level 1. This approximated version of the image is analysed.

Norm is a function that is used to measure the size of a vector. Norms of a vector x is given by: -

$$\|x\|_p = (\sum_i |x_i|^p)^{1/p}$$

1) It is called L^1 norm when $p = 1$,
2) It is called L^2 norm when $p = 2$
3) It is called Max norm when $p = \infty$

The L^1-norm also known as Manhattan distance is the sum of the magnitudes of the vectors in space. The L^1 norm of a signal $u \in L^1$ denoted by $\|u\|_1$ is determined as follows: -

$$\|u\|_1 = \int_0^\infty |u(t)|\, dt$$

The L^2 norm of a signal $u \in L^2$, denoted as $\|u\|_2$ is determined as follows: -

$$\|u\|_2 = \int_0^\infty (u(t)^2)^{1/2}\, dt$$

The L^2-norm is the square root of sum of the squares of magnitudes of the vectors in space. It represents the compactness of total energy contained in a signal.

5. Results and Discussion

We have selected an image of a girl as sample image (Figure 2).

Figure 2: Sample image and its wavelet decomposition

The analysis of approximated version of the given image using different wavelets by software MATLAB, are shown in figures 3, 4, 5, 6 and 7.

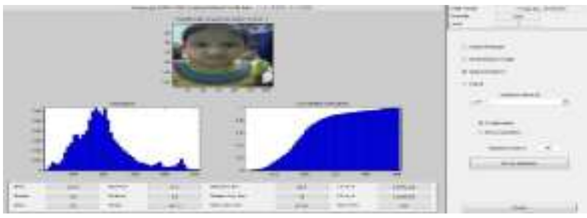

Figure 3: Approximation using Haar wavelet

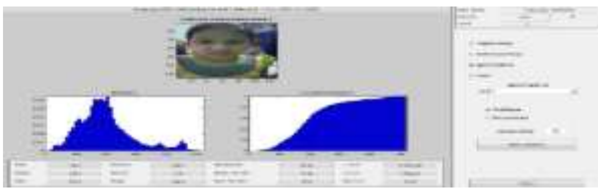

Figure 4: Approximation using symlet wavelet

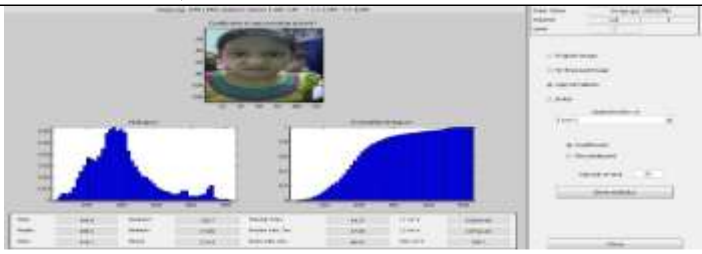

Figure 5: Approximation using coiflet wavelet

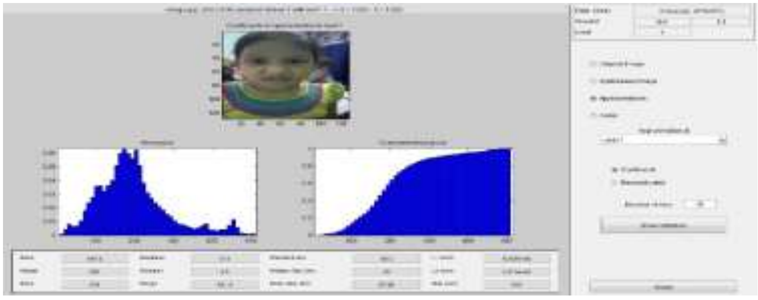

Figure 6: Approximation using biorthogonal wavelet

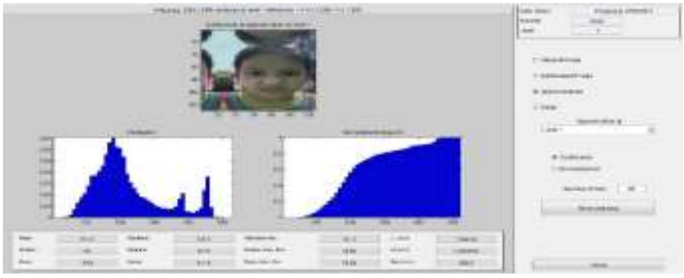

Figure 7: Approximation using discrete meyer wavelet

The L^1 *and* L^2-norms of approximated version of given image using different wavelets are enlisted in table 1.

Table 1: L^1 and L^2-norms using different wavelets

S. No.	Wavelet	L^1-norm	L^2-norm
1	Haar	9.707×10^6	4.814×10^4
2	Sym-2	9.797×10^6	4.860×10^4
3	Coif	9.825×10^6	4.873×10^4
4	Bior-1.1	9.707×10^6	4.814×10^4
5	Dmey	1.090×10^7	5.407×10^4

The value of L^1 *and* L^2-norms for the given image are lowest for Haar wavelet.

6. Conclusion

Wavelet transforms in image processing provides localized information of the image. Multiresolution analysis is a technique which has flexibility as per requirement of analysis. In wavelet transforms, the image is decomposed into four sub-images of one approximated version and three detail version after one level of decomposition. The L^1 *and* L^2-norms of the approximated version of image represents compactness of the energy. The minimum value of L^1 *and* L^2-norms are obtained by Haar wavelet, which indicates that Haar wavelet is the best adaptive wavelet for the wavelet analysis of given image.

References

1. Strang, G., 1993. Wavelet transforms versus Fourier transforms, *Bulletin of the American Mathematical Society*, Vol. 28(2), 288-305.
2. Antoine, J. P., 2004. Wavelet analysis: A new tool in Physics. *Wavelets in Physics*, Van Den Berg, J. C., ed., pp. 9-21.
3. Hernandez, E. and Weiss, G., 1996. *A First Course on Wavelets*, CRC Press, New York.
4. Goswami, J. C., and Chan, A.K., 1999. *Fundamentals of Wavelets Theory, Algorithms and Applications, Willey-Interscience, New York.*
5. Heil, C. and Walnut, D., 1989. Continuous and discrete wavelet transforms, *SIAM Review*, Vol. 31, 628-666.
6. Mallat, S.G., 1989. Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$, *Transactions of the American Mathematical Society*, Vol. 315, 69-87.
7. Mallat, S.G., 1989. A theory for multi resolution signal decomposition: the wavelet representation. *IEEE Transactions on Pattern Analysis and Machine Intelligence,* Vol.11(7), 674-693.
8. Daubechies, I., 1990. The wavelet transform, time frequency localization and signal analysis, *IEEE Transaction on Information Theory*, Vol. 6, 961–1005.
9. Kumar, A. and Kumari, M., 2016. Image compression by discrete wavelet transforms using global thresholding, *International Journal of Advance Research and Innovation*, Special issue, 1-5.
10. Singh, A.P., Potnis, A. and Kumar, A., 2016. A review on latest techniques of image compression, *International Research Journal of Engineering and Technology (IRJET)*, Vol. 3(7), 727-734.
11. Mustaffa, M. T., Hashim, N. M. Z., Saad, N. M., Hadi, N. A. A., Salleh, A. and Jaafar, A. S., 2015. Biomedical signal and image processing in MATLAB. *International journal for advance research in engineering and technology*, Vol. 3(9), pp. 29-32.